Stimulus-Response and the Fundamental Scale

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To be able to perceive and sense objects in the environment our brains miniaturize them within our system of neurons so that we have a proportional relationship between what we perceive and what is out there. Without proportionality we cannot coordinate our thinking with our actions with the accuracy needed to control the environment. Proportionality with respect to a single stimulus requires that our response to a proportionately amplified or attenuated stimulus we receive from a source should be proportional to what our response would be to the original value of that stimulus. If $w(s)$ is our response to a stimulus of magnitude s, then the foregoing gives rise to the functional equation $w(as) = b\, w(s)$. This equation can also be obtained as the necessary condition for solving the Fredholm equation of the second kind:

$$\int_a^b K(s,t)\, w(t)\, dt = \lambda_{\max} w(s)$$

obtained as the continuous generalization of the discrete formulation $Aw = \lambda_{\max} w$ for deriving priorities where instead of the positive reciprocal matrix A in the principal eigenvalue problem, we have a positive kernel, $K(s,t) > 0$, with $K(s,t)\, K(t,s) = 1$ that is also consistent i.e. $K(s,t)\, K(t,u) = K(s,u)$, for all s, t, and u . The solution of this functional equation in the real domain is given by

$$w(s) = Ce^{\log b \frac{\log s}{\log a}} P\left(\frac{\log s}{\log a}\right)$$

where P is a periodic function of period 1 and $P(0) = 1$. One of the simplest such examples with $u = \log s / \log a$ is $P(u) = \cos(u/2\pi)$ for which $P(0) = 1$.

The logarithmic law of response to stimuli can be obtained as a first order approximation to this solution through series expansions of the exponential and of the cosine functions as:

$$v(u) = C_1 e^{-\beta u} P(u) \approx C_2 \log s + C_3$$

$\log ab \equiv -\beta, \beta > 0$. The expression on the right is known as the Weber-Fechner law of logarithmic response $M = a \log s + b, a \neq 0$ to a stimulus of magnitude s. This law was empirically established and tested in 1860 by Gustav Theodor Fechner who used a law formulated by Ernest Heinrich Weber regarding discrimination between two nearby values of a stimulus. We have now shown that that Fechner's version can be derived by starting with a functional equation for stimulus response.

The integer-valued scale of response used in making paired comparison judgments can be derived from the logarithmic response function as follows. The larger the stimulus, the larger a change in it is needed for that change to be detectable. The ratio of successive just noticeable differences (the well-known "jnd" in psychology) is equal to the ratio of their corresponding successive stimuli values. Proportionality is maintained. Thus, starting with a stimulus s_0 successive magnitudes of the new stimuli take the form:

$$s_1 = s_0 + \Delta s_0 = s_0 + \frac{\Delta s_0}{s_0} s_0 = s_0(1+r)$$

$$s_2 = s_1 + \Delta s_1 = s_1(1+r) = s_0(1+r)^2 \equiv s_0 \alpha^2$$

$$\vdots$$

$$s_n = s_{n-1}\alpha = s_0 \alpha^n \,(n = 0,1,2,\ldots)$$

We consider the responses to these stimuli to be measured on a ratio scale (*b*=0). A typical response has the form $M_i = a \log \alpha^i$, i =1,…,*n*, or one after another they have the form:

$$M_1 = a \log \alpha, M_2 = 2a \log \alpha, ..., M_n = na \log \alpha$$

We take the ratios M_i / M_1, i = 1,…,*n*, of these responses in which the first is the smallest and serves as the unit of comparison, thus obtaining the *integer* values 1, 2, …, *n* of the fundamental scale of the AHP. It appears that numbers are intrinsic to our ability to make comparisons, and that they were not an invention by our primitive ancestors. We must be grateful to them for the discovery of the symbolism. In a less mathematical vein, we note that we are able to distinguish ordinally between high, medium and low at one level and for each of them in a second level below that also distinguish between high, medium and low giving us nine different categories. We assign the value one to (low, low) which is the smallest and the value nine to (high, high) which is the highest, thus covering the spectrum of possibilities between two levels, and giving the value nine for the top of the paired comparisons scale as compared with the lowest value on the scale. Because of increase in inconsistency when we compare more than about 7 elements, we don't need to keep in mind more than 7 ± 2 elements. This was first conjectured by the psychologist George Miller in the 1950's and explained in the AHP in the 1970's (Saaty and Ozdemir 2003a). Finally, we note that the scale just derived is attached to the importance we assign to judgments. If we have an exact measurement such as 2.375 and want to use it as it is for our judgment without attaching significance to it, we can use its entire value without approximation.

A person may not be schooled in the use of numbers and there are many in our world who do not, but still have feelings, judgments and understanding that enable him or her to make accurate comparisons (equal, moderate, strong, very strong and extreme and compromises between these intensities). Such judgments can be applied successfully to compare stimuli that are not too disparate but homogeneous in magnitude. By homogeneous we mean that they fall within specified bounds. The Fundamental Scale for paired comparisons, summarizes the foregoing discussion.

The idea of using time dependent judgments has been examined in detail and will not be discussed in this chapter (Saaty 2003).